ON THE EQUATIONS OF PLASTICITY FOR A CERTAIN LIMITING CONDITION

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In [1] equations of the theory of plasticity were proposed for an arbitrary relationship between the invariants of the stress tensor

$$F(s,\ t) = 0 \qquad (s = {}^1/_2(\sigma_x + \sigma_y),\ t = \sqrt{{}^1/_4(\sigma_x - \sigma_y)^2 + \tau_{xy}^2})$$

In accordance with this work, we represent the relationship between s and t in the form of a relationship between the functions ψ and λ, which are defined as follows:

$$dt\,/\,ds = \cos 2\psi, \qquad 2d\lambda = \tan 2\psi\, d\ln t \tag{1}$$

The canonical system of characteristic equations in the xy plane takes on the form:

$$\frac{\partial y}{\partial \xi} = \tan(\varphi - \psi)\frac{\partial x}{\partial \xi} \qquad \frac{\partial y}{\partial \eta} = \tan(\varphi + \psi)\frac{\partial x}{\partial \eta} \qquad (\xi = \lambda(\psi) + \varphi,\ \eta = \lambda(\psi) - \varphi) \tag{2}$$

The canonical system of characteristic equations in the uv plane takes on the form:

$$\frac{\partial u}{\partial \xi} = -\tan(\varphi - \psi)\frac{\partial v}{\partial \xi}, \qquad \frac{\partial u}{\partial \eta} = -\tan(\varphi + \psi)\frac{\partial v}{\partial \eta} \tag{3}$$

In [1,2] it was shown that the systems of Equations (2) and (3) are integrable in closed form when $\psi(\lambda) = \lambda$; in the present work a solution of the systems (2) and (3)is given for the case $\psi(\lambda) = \lambda/3$. We examine the system (2). Introducing the new functions

$$X = -x\sin(\varphi + \psi) + y\cos(\varphi + \psi), \qquad Y = x\cos(\varphi + \psi) + y\sin(\varphi + \psi) \tag{4}$$

and substituting into the system (2), we obtain

$$\frac{\partial X}{\partial \eta}+\frac{\psi'(\lambda)-1}{2}Y=0; \qquad \cot 2\psi\frac{\partial X}{\partial \xi}-\frac{1+\psi'(\lambda)}{2}X+\frac{\partial Y}{\partial \xi}+\cot 2\psi\frac{1+\psi'(\lambda)}{2}Y=0 \tag{5}$$

Upon eliminating the function Y we arrive at the equation

$$\frac{\partial^2 X}{\partial \eta \partial \xi}+\left(\frac{\psi''}{2(1-\psi')}+\cot 2\psi\frac{1+\psi'}{2}\right)\frac{\partial X}{\partial \eta}+\cot 2\psi\frac{1-\psi'}{2}\frac{\partial X}{\partial \xi}-\frac{1-\psi'^2}{4}X=0 \tag{6}$$

In the case of $\psi(\lambda) = \lambda/3$, one of the Laplace invariants [3] of Equation (6) vanishes. Therefore, Equation (6) can be integrated in closed form:

$$X(\xi,\ \eta)=\operatorname{cosec}^2\frac{\xi+\eta}{3}\left(F_0(\eta)+\int_{\xi_0}^{\xi}F_1(\zeta)\sin\frac{\zeta+\eta}{3}\,d\zeta\right) \tag{7}$$

Here $F_0(\eta)$ and $F_1(\zeta)$ are arbitrary functions. We obtain the expression for the function $Y(\xi,\ \eta)$ from the first of the equations of the system (5).

We turn now to an investigation of the velocity field. We introduce the functions

$$U=-u\sin(\varphi-\psi)+v\cos(\varphi-\psi), \qquad V=u\cos(\varphi-\psi)+v\sin(\varphi-\psi) \tag{8}$$

Then the equations of the system (3) take on the form:

$$\frac{\partial U}{\partial \eta}-\cot 2\psi\frac{1+\psi'(\lambda)}{2}U-\cot 2\psi\frac{\partial V}{\partial \eta}+\frac{1+\psi'(\lambda)}{2}V=0, \qquad \frac{\partial V}{\partial \xi}+\frac{1-\psi'(\lambda)}{2}U=0 \tag{9}$$

Eliminating the function U, we obtain

$$\frac{\partial^2 V}{\partial \xi \partial \eta}+\left(-\frac{\psi''}{2(1-\psi')}+\cot 2\psi\frac{1+\psi'}{2}\right)\frac{\partial V}{\partial \xi}-\cot 2\psi\frac{1-\psi'}{2}\frac{\partial V}{\partial \eta}+\frac{1-\psi'^2}{4}V=0 \tag{10}$$

In the case of $\psi(\lambda) = \lambda/3$ one of the Laplace invariants of Equation (10) is zero. Therefore the solution has the form

$$V(\xi,\ \eta)=\operatorname{cosec}^2\frac{\xi+\eta}{3}\left(F_2(\xi)+\int_{\eta_0}^{\eta}F_3(\zeta)\sin\frac{\zeta+\xi}{3}\,d\zeta\right) \tag{11}$$

We remark that other integrable cases of Equations (6) and (10) can be obtained from the condition that the Laplace invariant of the equations vanish. This condition is obtainable from (6) and (10) by means of the Laplace transformation [3]. In all cases, the function $\psi'' = 0$ will be a solution of the corresponding nonlinear equations. However, in this process the expression of the general integral becomes quite complicated.

We turn to the condition for the limiting state in the case $\psi(\lambda) = \lambda/3$.

From Equations (1) we obtain the parametric representation

$$t = k(\sin 2\psi)^3, \qquad s = 3k(\psi - {}^1/_4 \sin 4\psi) + m \qquad (k,\ m = \text{const}) \tag{12}$$

In plane strain, the condition of the limiting state is also represented by an envelope of stress circles. Then the condition being proposed is written in the form

$$\sigma_n = k(3\psi - \sin 4\psi + {}^1/_8 \sin 8\psi) + m$$
$$|\tau| = {}^1/_8 k(3 + \cos 8\psi - 4\cos 4\psi) \tag{13}$$

In Fig. 1 is shown the envelope to the stress circle obtained by von Karman [5] from an experiment with sandstone. This is seen to be extremely close to the curve (13) for k = 1900 kg/cm^2 and m = - 8850 kg/cm^2.

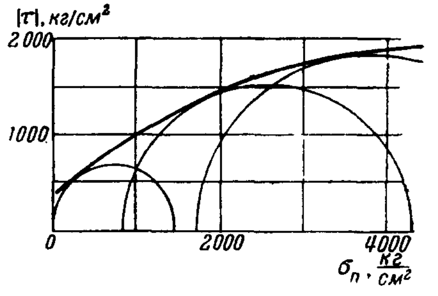

Fig. 1.

An analysis of cases in which at least one of the variables ξ or η is constant in the region being examined can be carried out as indicated in [1]. Let $\xi = \xi_0$ = const. Then the family of characteristics consists of the straight lines

$$y = x\tan(\varphi - \psi) + \Phi(\eta) = x \tan {}^1/_3 (\xi_0 - 2\eta) + \Phi(\eta) \tag{14}$$

Here $\Phi(\eta)$ is an arbitrary function. The second family is found by the integration of the equation

$$dy / dx = \tan(\varphi + \psi) = \tan {}^1/_3 (2\xi_0 - \eta) \tag{15}$$

consistent with (14). For $\eta = \eta_0$ = const the investigation is analogous. When both quantities are constant, each of the families of characteristics consists of parallel straight lines. As an example we examine the compression of a strip of height $2b$ which is loaded by a uniform pressure p distributed on oppositely lying segments of length a (Fig. 2). It is convenient to take the origin of coordinates at the point B. In the triangle ABC there is a homogeneous state of stress. In the region ACD the family of characteristics η = const is a pencil of straight lines passing through the point A:

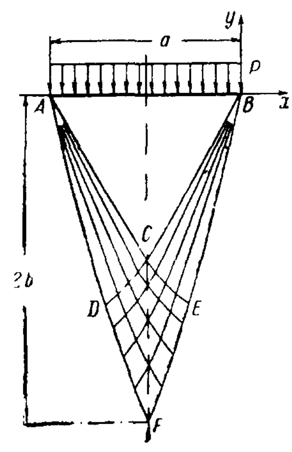

Fig. 2.

$$y = (x + a) \tan {}^1/_3 (\xi_0 - 2\eta) \tag{16}$$

From (16) and (15) we obtain the differential equation for the characteristics of the second family

$$\frac{dy}{dx} = \tan\left[\frac{1}{2}\left(\tan^{-1}\frac{y}{x+a} + \xi_0\right)\right] \tag{17}$$

Hence, by integration, we obtain the parametric representation of these characteristics

$$\begin{gathered} x(\xi_0, \eta) = C\frac{\sqrt{1+\tan^2 {}^1/_3(\xi_0 - 2\eta)} - \tan {}^1/_3(\xi_0 - 2\eta)}{\left[\sqrt{1+\tan^2 {}^1/_3(\xi_0 - 2\eta)} - \tan {}^1/_3(\xi_0 - 2\eta) - \dfrac{1-\tan {}^1/_2\,\xi_0}{1+\tan {}^1/_2\,\xi_0}\right]^2} - a \\ y(\xi_0, \eta) = (x+a)\tan {}^1/_3(\xi_0 - 2\eta) \end{gathered} \tag{18}$$

For the characteristic *CD*, the constant *C* is determined from the condition that $x = -a/2$ at $\eta = \eta_0$. In an analogous fashion we obtain the parametric representation for the characteristics in the region *BCE*. Omitting the intermediate calculations, we have as the expressions for the functions $X(\xi, \eta)$ on the characteristics *CD* and *CE*

$$\begin{gathered} X(\xi_0, \eta) = -x(\xi_0, \eta)\sin {}^1/_3(2\xi_0 - \eta) + [x(\xi_0, \eta) + a]\tan {}^1/_3(\xi_0 - 2\eta)\cos {}^1/_3(2\xi_0 - \eta) \\ X(\xi, \eta_0) = 0 \end{gathered} \tag{19}$$

Therefore we obtain the solution in the region *CDEF*

$$X(\xi, \eta) = \operatorname{cosec}^2 {}^1/_3(\xi + \eta)\sin^2 {}^1/_3(\xi_0 + \eta)\, X(\xi_0, \eta) \tag{20}$$

It is further necessary to determine the resulting stress σ_{x_2} along the vertical axis of symmetry and to obtain the sought relationship between *a*, *b* and *p* from the vanishing condition.

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